Exercises 13, 20.05.2025, Solutions

Exercise 1.

In order to solve the problem, we introduce the following complex notations for pressure and thickness change:

$$\tilde{p} = p_0 e^{i\omega t},\ \widetilde{\Delta L} = \Delta L_0 e^{i(\omega t+\varphi)}.$$

The real part of these variables describes the real applied pressure and measured thickness change, as defined in the problem:

$$\mathrm{Re}[\tilde{p}] = \mathrm{Re}\left[p_0 e^{i\omega t}\right] = p_0 \cos(\omega t), \qquad \mathrm{Re}\left[\widetilde{\Delta L}\right] = \mathrm{Re}\left[\Delta L_0 e^{i(\omega t+\varphi)}\right] = \Delta L_0 \cos(\omega t + \varphi).$$

Besides, it is possible to show that expression $\left|\frac{L\tilde{p}}{\widetilde{\Delta L}}\right|$ gives indeed the Young modulus:

$$\left|\frac{L\tilde{p}}{\widetilde{\Delta L}}\right| = \left|\frac{Lp_0 e^{i\omega t}}{\Delta L_0 e^{i(\omega t+\varphi)}}\right| = \left|\frac{Lp_0}{\Delta L_0} e^{-i\varphi}\right| = \frac{Lp_0}{\Delta L_0} = Y(\omega).$$

In the following, we will also use complex designations for all variables in the problem ($\widetilde{E_3}$, $\widetilde{D_3}$, $\widetilde{\sigma_3}$, $\widetilde{\varepsilon_3}$), real values of which give the real measured values in the sample:

$$\mathrm{Re}\left[\widetilde{E_3}\right] = E_3, \qquad \mathrm{Re}\left[\widetilde{D_3}\right] = D_3, \qquad \mathrm{Re}\left[\widetilde{\sigma_3}\right] = \sigma_3, \qquad \mathrm{Re}\left[\widetilde{\varepsilon_3}\right] = \varepsilon_3.$$

In the geometry of the problem, the values for deformation ε_3 and the stress σ_3 are defined as:

$$\varepsilon_3 = \frac{\Delta L}{L}, \qquad \sigma_3 = -p,$$

and the same relationships are valid in complex designations as well:

$$\widetilde{\varepsilon_3} = \frac{\widetilde{\Delta L}}{L}, \qquad \widetilde{\sigma_3} = -\tilde{p},$$

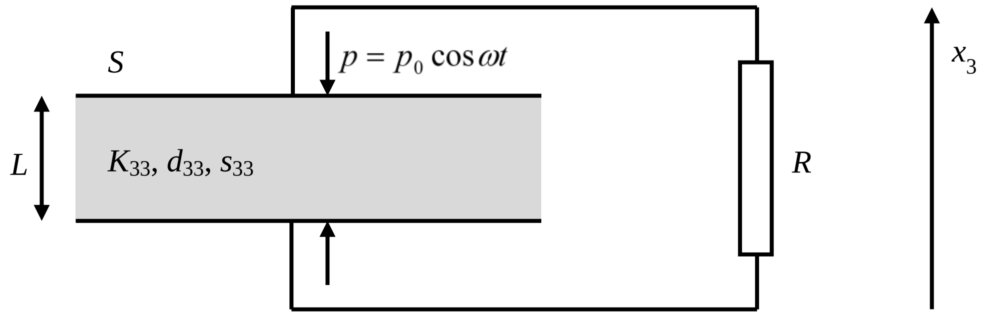
In the new designations, the expression for Young modulus attains the following form:

$$Y(\omega) = \left|\frac{\tilde{p}}{\widetilde{\Delta L}/L_0}\right| = \left|\frac{-\widetilde{\sigma_3}}{\widetilde{\varepsilon_3}}\right| = \left|\frac{\widetilde{\sigma_3}}{\widetilde{\varepsilon_3}}\right|$$

Thus, in order to obtain the frequency dependence of the Young modulus, we have to find the relation between $\widetilde{\sigma_3}$ and $\widetilde{\varepsilon_3}$. To do it, we will use the constitutive equations for constant temperature in complex variables:

$$\widetilde{D_3} = \varepsilon_0 K_{33} \widetilde{E_3} + d_{33}\widetilde{\sigma_3},$$
$$\widetilde{\varepsilon_3} = d_{33}\widetilde{E_3} + s_{33}\widetilde{\sigma_3}.$$

Since the system is neither short-circuited nor open-circuited, both $\widetilde{D_3}$ and $\widetilde{E_3}$ are not zero. It is possible though to obtain relation between them from the theory of electricity.



In the following, we first consider the equivalent electric scheme shown in the figure above and obtain relation between E_3 and D_3 in real variables; after that we will generalize the result to complex variables.

Let us suppose that there is an electric displacement D_3 inside the sample. From electrostatics, this electric displacement generates free charges on the electrodes Q, having value

$$Q = D_3 S,$$

where S is the surface area of the electrodes. Since the plates of the capacitance are joined by the resistance R, the charge leaks through the resistance with the current I:

$$I = -\frac{dQ}{dt} = S\frac{dD_3}{dt}.$$

In turn, the current I, by Ohm's law, generates the potential difference V on the resistance:

$$V = -IR = SR\frac{dD_3}{dt}.$$

Finally, from electrostatics, the electric field inside the sample can be found as

$$E_3 = \frac{V}{L} = \frac{SR}{L}\frac{dD_3}{dt},$$

which is the sought relation between electric field E_3 and electric displacement D_3. In complex variables, this relation has the same form:

$$\widetilde{E_3} = \frac{SR}{L}\frac{d\widetilde{D_3}}{dt}.$$

It is obvious that, since all effects in the system are generated by the application of pressure $p = p_0 \cos \omega t$ with frequency ω, the electric displacement D_3 can be represented in the form $D_3 = D_0 \cos(\omega t + \psi)$, also changing with frequency ω. Then, in complex notations, the electric displacement has the form

$$\widetilde{D_3} = D_0 e^{i(\omega t+\psi)},$$

and the derivative on the electric displacement can be easily found as follows:

$$\frac{d\widetilde{D_3}}{dt} = i\omega\widetilde{D_3}.$$

This allows us to rewrite the relation between $\widetilde{E_3}$ and $\widetilde{D_3}$:

$$\widetilde{E_3} = \frac{i\omega SR}{L}\widetilde{D_3}, \qquad \widetilde{D_3} = \frac{L}{i\omega SR}\widetilde{E_3}.$$

We substitute this relation into the constitutive equation for electric displacement:

$$\widetilde{D_3} = \varepsilon_0 K_{33}\widetilde{E_3} + d_{33}\widetilde{\sigma_3},$$

$$\frac{L}{i\omega SR}\widetilde{E_3} = \varepsilon_0 K_{33}\widetilde{E_3} + d_{33}\widetilde{\sigma_3},$$

$$\widetilde{E_3} = \frac{d_{33}\widetilde{\sigma_3}}{\frac{L}{i\omega SR} - \varepsilon_0 K_{33}} = -\frac{d_{33}}{\varepsilon_0 K_{33}}\frac{1}{1 - \frac{1}{i\omega RC}}\widetilde{\sigma_3} = -\frac{d_{33}}{\varepsilon_0 K_{33}}\frac{\omega}{\omega + i\omega_0}\widetilde{\sigma_3},$$

where $C = (\varepsilon_0 K_{33} S/L)$ is the capacitance of the sample, $\omega\_0 = (1/RC)$ is the characteristic frequency of the sample. Then, we substitute the obtained expression for $\widetilde{E_3}$ into the constitutive equation for the deformation:

$$\widetilde{\varepsilon_3} = d_{33}\widetilde{E_3} + s_{33}\widetilde{\sigma_3},$$

$$\widetilde{\varepsilon_3} = d_{33}\left(-\frac{d_{33}}{\varepsilon_0 K_{33}}\frac{\omega}{\omega + i\omega_0}\widetilde{\sigma_3}\right) + s_{33}\widetilde{\sigma_3},$$

$$\widetilde{\varepsilon_3} = \left(s_{33} - \frac{d_{33}^2}{\varepsilon_0 K_{33}}\frac{\omega}{\omega + i\omega_0}\right)\widetilde{\sigma_3},$$

$$\frac{\widetilde{\sigma_3}}{\widetilde{\varepsilon_3}} = \frac{1}{s_{33} - \frac{d_{33}^2}{\varepsilon_0 K_{33}} \frac{\omega}{\omega + i\omega_0}}.$$

1. For the case $\omega \ll \omega_0$, we can simply put $\omega = 0$ in this expression, and

$$\left(\frac{\widetilde{\sigma_3}}{\widetilde{\varepsilon_3}}\right)_{\omega \ll \omega_0} = \frac{1}{s_{33}},$$

$$(Y)_{\omega \ll \omega_0} = \frac{1}{s_{33}},$$

which is indeed the Young modulus measured for the short-circuited geometry.

2. For the case $\omega \gg \omega_0$, we put $\frac{\omega}{\omega + i\omega_0} = 1$, and

$$\left(\frac{\widetilde{\sigma_3}}{\widetilde{\varepsilon_3}}\right)_{\omega \gg \omega_0} = \frac{1}{s_{33} - \frac{d_{33}^2}{\varepsilon_0 K_{33}}},$$

$$(Y)_{\omega \gg \omega_0} = \frac{1}{s_{33} - \frac{d_{33}^2}{\varepsilon_0 K_{33}}},$$

which is indeed the Young modulus measured for the open-circuited geometry.